

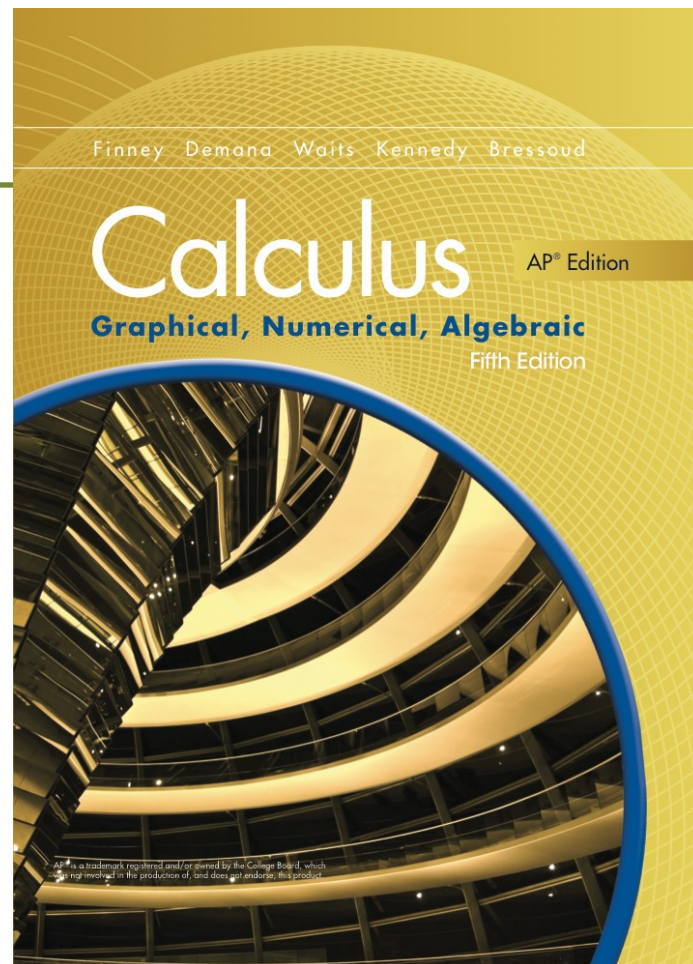
Chapter 7

Differential Equations and Mathematical Modeling



Section 7.4

Exponential Growth and Decay



Quick Review

1. Rewrite the equation in exponential form or logarithmic form.
 - a. $\ln a = b$
 - b. $e^c = d$
2. Solve for x
 - a. $\ln(x+2) = 3$
 - b. $2e^x = 10$
 - c. $1.5^x = 2$
3. Solve for y given $\ln(y+1) = 2x - 3$.

Quick Review Solutions

1. Rewrite the equation in exponential form or logarithmic form.

a. $\ln a = b$ $e^b = a$

b. $e^c = d$ $\ln d = c$

2. Solve for x

a. $\ln(x+2) = 3$ $x = e^3 - 2$

b. $2e^x = 10$ $x = \ln 5$

c. $1.5^x = 2$ $x = \frac{\ln 2}{\ln 1.5}$

3. Solve for y given $\ln(y+1) = 2x - 3$. $y = e^{2x-3} - 1$

What you'll learn about

- The differential equation $dy/dt = ky$ and the law of exponential change
- Continuously compounded interest
- Radioactive decay
- Modeling growth in convenient bases
- Newton's Law of Cooling

... and why

Understanding the differential equation $dy/dt = ky$ gives us new insight into exponential growth and decay.

Separable Differential Equation

A differential equation of the form $\frac{dy}{dx} = f(y) g(x)$ is called **separable**. We **separate the variables** by writing it in the form

$$\frac{1}{f(y)} dy = g(x) dx$$

The solution is found by antidiifferentiating each side with respect to its thusly isolated variable.

Example Solving by Separation of Variables

Solve for y if $\frac{dy}{dx} = x^2 y^2$ and $y=3$ when $x=0$.

$$\frac{dy}{dx} = x^2 y^2$$

$$y^2 dy = x^2 dx$$

$$\int y^2 dy = \int x^2 dx$$

$$- y^{-1} = \frac{x^3}{3} + C$$

Apply the initial conditions to find C .

$$- \frac{1}{3} = C \text{ So, } - y^{-1} = \frac{x^3}{3} - \frac{1}{3} \text{ and } y = \frac{3}{1 - x^3}$$

This solution is valid for the continuous section of the function that goes through the point $(0, 3)$, that is, on the domain $(-\infty, 1)$.

The Law of Exponential Change

If y changes at a rate proportional to the amount present

(that is, if $\frac{dy}{dt} = ky$), and if $y = y_0$ when $t = 0$, then

$$y = y_0 e^{kt}$$

The constant k is the **growth constant** if $k > 0$ or the **decay constant** if $k < 0$.

Continuously Compounded Interest

If the interest is added continuously at a rate proportional to the amount in the account, you can model the growth of the account with the initial value problem:

Differential equation: $\frac{dA}{dt} = rA$

Initial condition: $A(0) = A_0$

The amount of money in the account after t years at an annual interest rate r :

$$A(t) = A_0 e^{rt}.$$

Example Compounding Interest Continuously

Suppose you deposit \$500 in an account that pays 5.3% annual interest. How much will you have 4 years later if the interest is **(a)** compounded continuously? **(b)** compounded monthly?

Let $A_0 = 500$ and $r = 0.053$.

a. $A(4) = 500e^{(0.053)(4)} = 618.07$

b. $A(4) = 500\left(1 + \frac{0.053}{12}\right)^{(12)(4)} = 617.79$

Example Finding Half-Life

Find the half-life of a radioactive substance with decay equation

$$y = y_0 e^{kt}.$$

The half-life is the solution to the equation $y_0 e^{kt} = \frac{1}{2} y_0$.

Solve algebraically

$$\begin{aligned} e^{kt} &= \frac{1}{2} \\ -kt &= \ln \frac{1}{2} \\ t &= -\frac{1}{k} \ln \frac{1}{2} = \frac{\ln 2}{k} \end{aligned}$$

Note: The value t is the half-life of the element. It depends only on the value of k .

Half-life

The **half - life** of a radioactive substance with rate constant k ($k > 0$) is

$$\text{half-life} = \frac{\ln 2}{k}.$$

Newton's Law of Cooling

The rate at which an object's temperature is changing at any given time is roughly proportional to the difference between its temperature and the temperature of the surrounding medium. If T is the temperature of the object at time t , and T_s is the surrounding temperature, then

$$\frac{dT}{dt} = -k(T - T_s). \quad (1)$$

Since $dT = d(T - T_s)$, rewrite (1)

$$\frac{d}{dt}(T - T_s) = -k(T - T_s)$$

Its solution, by the law of exponential change, is

$$T - T_s = (T_0 - T_s) e^{-kt},$$

Where T_0 is the temperature at time $t = 0$.

Example Using Newton's Law of Cooling

A temperature probe is removed from a cup of coffee and placed in water that

has a temperature of $T_s = 4.5^\circ\text{C}$.

Temperature readings T , as recorded in the table below, are taken

after 2 sec, 5 sec, and every 5 sec thereafter.

Table 7.1 Experimental Data

Estimate

(a) the coffee's temperature at the time 2
the temperature probe was removed,

(b) the time when the temperature
probe reading will be 8°C .

Time (sec)	T ($^\circ\text{C}$)	$T - T_s$ ($^\circ\text{C}$)
2	64.8	60.3
5	49.0	44.5
10	31.4	26.9
15	22.0	17.5
20	16.5	12.0
25	14.2	9.7
30	12.0	7.5

Example Using Newton's Law of Cooling

According to Newton's Law of Cooling, $T - T_s = (T_0 - T_s)e^{kt}$, where $T_s = 4.5$ and T_0 is the temperature of the coffee at $t = 0$.

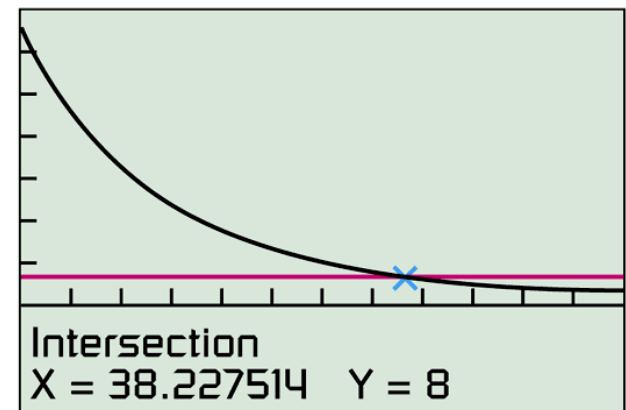
Use exponential regression to find that $T - 4.5 = 61.55(0.9277)^t$ is a model for the $(t, T - T_s) = (t, T - 4.5)$ data. Thus

$T = 4.5 + 61.66(0.9277)^t$ is a model of the (t, T) data.

(a) At time $t = 0$ the temperature was

$$T = 4.5 + 61.66(0.9277) \approx 66.16^\circ \text{C}$$

(b) The figure below shows the graphs of $y = 8$ and $y = T = 4.5 + 61.66(0.9277)^t$



$[0, 60]$ by $[-20, 70]$

(b)